Application:

Any group of order ≤ 59 is solvable.

(Note $|A_5| = 60$, A_5 is non-solvable).

Basic observation:

$$|G| = n = p_1^{r_1} \cdots p_s^{r_s}$$

For each $p \in \{p_1 \cdots p_s\}$. write N_p = number of sylow p-subgps.

By sylow's theorem,

$$N_p = pl + 1$$

and $N_p \mid (n/p^r)$, say $p = p_i$, $r = r_i$.

(This is because $N_p = |G/N_H|$, where $H \leq G$ a sylow-p subgp. For $H \leq G$, we have $H \leq N_H \leq G$)

Fact 1: p-group is solvable

Fact 2: $|G| = p \cdot q$, $p \neq q$ $\Rightarrow$ G is solvable

pf: WLOG, assume $p < q$. By the above observation,

$$N_q = q \cdot l + 1 \mid p \quad \Rightarrow l = 0$$

i.e. Sylow q-subgp is unique, and hence normal

Denote $H_q \trianglelefteq G$ the unique sylow q-subgp.

Then $|G/H_q| = p \Rightarrow G/H_q \simeq \mathbb{Z}/p\mathbb{Z}$ is solvable

$\Rightarrow G$ is solvable #

Fact 3 $|G| = p^2 q$, $p \neq q$. $\Rightarrow G$ is solvable

pf: Case(i) $p > q$

Consider sylow p-subgp.: $N_p = pl+1 \mid q$

$\Rightarrow N_p = 1$

Case(ii) $p < q$

$N_q = ql+1 \mid p^2$

$\Rightarrow N_q = 1, p, p^2$

$N_q = 1 \Rightarrow$ okay

$N_q = p$: $\left[G \xrightarrow{\phi} S_p \right.$ the hom. given by the conjugation on all sylow q-subgps

$\phi \neq \{e\}$ (in fact, $\phi(G)$ contains a p-cycle; why?)

$q > p \Rightarrow \ker(\phi) \neq \{e\}$

and $\left| \frac{G}{\ker(\phi)} \right| = p$-power

$\Rightarrow \frac{G}{\ker \phi}$ solvable $\left. \right]$

Remark: This case is impossible!

Since $|\ker \phi| = q \cdot p$ or $q \Rightarrow \ker \phi$ is solvable $\Rightarrow G$ solvable

$N_q = p^2$:

Consider the elements coming from all sylow q-gps :

Note : $\{H_i\}_{1 \le i \le p^2}$ $\quad H_i \le G$ sylow q-subgp

$$H_i \cap H_j = \{e\}, \quad i \neq j$$

$$\Rightarrow \bigcup_{i=1}^{p^2} H_i = \left(\coprod_{i=1}^{p^2} H_i - \{e\} \right) \amalg \{e\}.$$

So we get $p^2(q-1)+1$ elements in total

$$p^2 q - \left(p^2(q-1)+1\right) = p^2 - 1.$$

$\Rightarrow$ Sylow p-subgp in G is unique, and have normal

$\Rightarrow$ okay.

#

List:

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	(24)	25	26	27	28	29	(30)
31	32	33	34	35	(36)	37	38	39	(40)
41	(42)	43	44	45	46	47	(48)	49	50
51	52	53	(54)	55	(56)	57	58	59	

Case 1: 30, 42
$\|$ $\|$
$2\times3\times5$ $2\times3\times7$

30: $N_5 = 5\ell+1 \mid 6$

If $N_5 = 6$, then $6\times(5-1)+1 = 25$

$\Rightarrow$ the remaining $30-25=5$ elements have the all non-unit elt of sylow 3-subgps

$\Rightarrow$ The has the unique sylow 3-subgp

$\Rightarrow$ okay

42: $N_7 = 7\ell+1 \mid 6 \Rightarrow N_7 = 1$ $\Rightarrow$ okay.

Case 2: 24, 40, 48, 56
$\|$ $\|$ $\|$ $\|$
$2^3\times3$ $2^3\times5$ $2^4\times3$ $2^3\times7$

24, 48: Just consider the case $N_2 \neq 1$. $\Rightarrow N_2 = 3$

$\Rightarrow G \xrightarrow{\phi} S_3$ $|\ker\phi| = 2^r$, + all subgps of S_3 are solvable

$\Rightarrow G$ solvable

40: $N_5 = 5\ell+1 \mid 8 \Rightarrow N_5 = 1$ $\Rightarrow$ okay

$(p, n) = 1$

$= p^{r_1}$

otherwise, we're done.

$+ r \leq p^{r_1} = \text{ord}(x)$

$\Rightarrow$ done

Then $p^r \tilde{y} \in A_1$

i.e $p^r \tilde{y} = (p^k \cdot n) x$,

Then $\text{ord}(nx) = \text{ord}(x)$

We can assume $k < r_1$.

Set $x' = nx$.

Then $\text{ord}(\tilde{y}) = p^{r_1 - k}$

$\Rightarrow r_1 - k + r_0 \leq r_1$

#

Lecture 7 Finitely generated abelian groups.

S = set

The free abelian group, generated by S, is the following group

$\in \mathbb{Z}$, and only finitely many n_i's are non-zero $\}$

$$\mathbb{Z}\langle S\rangle = \{ \sum_{x_i \in S} n_i x_i \mid n$$

on

In particular, if $|S| < +\infty$, t

$$\mathbb{Z}^{|S|}$$

$$\mathbb{Z}\langle S\rangle \cong \underbrace{\mathbb{Z} \times \mathbb{Z} \times}_{r=|S|}$$

$$\downarrow \qquad \downarrow$$

$$\sum_{i=1}^{r} n_i x_i \longmapsto (n_1, n_2,$$

$\cdots, n_r)$

Then

Prop: $|S| = r$, $|T| = S$.

iff $r = S$

$$\mathbb{Z}\langle S\rangle \cong \mathbb{Z}\langle T\rangle$$

pf: ($\Leftarrow$) if $r = S$, then

$$\mathbb{Z}^S \cong \mathbb{Z}\langle T\rangle$$

$$\mathbb{Z}\langle S\rangle \cong \mathbb{Z}$$

($\Rightarrow$) Assume $\mathbb{Z}^r \overset{\phi}{\cong} \mathbb{Z}^S$.

Then take any prime number p.

Consider $p\mathbb{Z}^r = \{ p(n_1, n_2, \cdots, n_r) = (pn_1, \cdots, pn_r) \in \mathbb{Z}^r \} \leq \mathbb{Z}^r$

Then $\phi(p\mathbb{Z}^r) = p\mathbb{Z}^S$

And $\dfrac{\mathbb{Z}^r}{p\mathbb{Z}^r} \overset{\bar{\varphi}}{\cong} \dfrac{\mathbb{Z}^s}{p\mathbb{Z}^s}$

$\| \qquad\qquad \|$

$\underbrace{\dfrac{\mathbb{Z}}{p\mathbb{Z}} \times \cdots \times \dfrac{\mathbb{Z}}{p\mathbb{Z}}}_{r} \qquad \underbrace{\dfrac{\mathbb{Z}}{p\mathbb{Z}} \times \cdots \times \dfrac{\mathbb{Z}}{p\mathbb{Z}}}_{s}$

$\Rightarrow \left|\dfrac{\mathbb{Z}^r}{p\mathbb{Z}^r}\right| = \left|\dfrac{\mathbb{Z}^s}{p\mathbb{Z}^s}\right| \Rightarrow r = s.$

$\| \qquad\qquad \|$

$p^r \qquad p^s$

#

The number r is called the rank of a finitely generated free abelian group.

Def: A : abelian group. Then

$$A_{tor} = \{x \in A \mid \operatorname{ord}(x) < +\infty\} \leq A$$

is called the torsion subgroup of A.

Warning : $\{x \in A \mid \operatorname{ord}(x) = +\infty\} \cup \{0\} \subseteq A$ is in general not a subgroup.

Structure Theorem of f.g abelian groups :

Let A be a f.g abelian group. Then

(1) $\exists$ a free abelian subgroup $A_{free} \leq A$, s.t

$$A = A_{free} \oplus A_{tor},$$

(A_{free} may not be unique)

where A_{free} and A_{tor} are both finitely generated.

(hence A_{free} are determined by its rank, and

A_{tor} is a finite abelian group)

$\Rightarrow$ $G \longrightarrow S_3$ $|\ker\phi| = 2$, + all subgps of S_3 are solvable

$\Rightarrow$ G is solvable

(2) Assume A is a finite abelian group.

Then (i) $A = \bigoplus_p A(p)$, where $A(p)$ is the sylow p-subgp of A.

(ii) For each finite abelian p-group $A(p)$.

Tha $\exists$ uniquely determined tuples

$$(p^{r_1}, p^{r_2} \ldots, p^{r_s}) \text{ with}$$

$$r_1 \geq r_2 \geq \ldots \geq r_s \geq 1$$

s.t $A(p) \cong \bigoplus_{i=1}^{s} \mathbb{Z}/p^{r_i}\mathbb{Z}$

Proof of (2) (ii):

(Existence)

Take an elt $x \in A$. s.t

$\operatorname{ord}(x)$ is maximal

Say $\operatorname{ord}(x) = p^{r_1}$

Set $A_1 = \langle x \rangle$. and consider

$$A \xrightarrow{\pi} A/A_1$$

Claim 1. $\forall\ \bar{y} \in A/A_1$, $\exists\ y \in A$, s.t

$$\operatorname{ord}(y) = \operatorname{ord}(\bar{y}).$$

Assume the claim1:

By induction on $|A|$,

$$\frac{A}{A_1} \cong \bigoplus_{i=2}^{s} \mathbb{Z}/p^{r_i}, \qquad r_2 \geq r_3 \geq \cdots \geq r_s \geq 1$$

By the claim1, $\exists\ x_2, x_3, \ldots, x_s \in A$, s.t

$$\langle x_i \rangle \cong \mathbb{Z}/p^{r_i}\mathbb{Z}, \quad \forall\ 2 \leq i \leq s.$$

Claim 2: $A = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_s \rangle$

$$r_1 + \cdots + r_s = r_1' + \cdots + r_{s'}'$$

Consider $\frac{A}{pA}$. Then

$$\text{LHS} \cong \frac{\mathbb{Z}}{p^{r_1-1}\mathbb{Z}} \times \cdots \times \frac{\mathbb{Z}}{p^{r_s-1}\mathbb{Z}}$$

$$\text{RHS} \cong \frac{\mathbb{Z}}{p^{r_1'-1}\mathbb{Z}} \times \cdots \times \frac{\mathbb{Z}}{p^{r_{s'}'-1}\mathbb{Z}}$$

$$\Rightarrow \sum_{i=1}^{s} r_i - s = \sum_{i=1}^{s'} r_i' - s'$$

$$\Rightarrow \quad s = s'.$$

Compare r_1 and r_2

If $r_1 > r_1'$ (or $r_1 < r_1'$), then consider

$$\frac{A}{p^{r_1'}A}.$$

Then LHS $\neq 0$, RHS $= 0$ ↯

Thus $r_1 = r_1'$

Case (1): $r_1 = r_2 = \cdots = r_s$, $r_1' = r_2' = \cdots = r_s'$

we're done